Tinbergen Institute

Fundamental Mathematics

Linear Algebra Preparatory Test 2

Question 1

Let

$$A = \begin{pmatrix} 5 & 9 & -1 & 15 \\ -3 & 1 & 7 & -4 \\ 1 & 2 & 0 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix},$$

and let T denote the linear transformation $T(x) = Ax$. Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that the image of S is $\mathbb{R}^2$ and the null space of S is $\{x \in \mathbb{R}^3 \ : \ x = \lambda b, \ \lambda \in \mathbb{R}\}$. Let ST denote the composition of the linear transformations S and T.

a. Find a basis for the null space of T.

b. Find the image of T, and show that b is an element of the image.

c. Find all vectors x such that $T(x) = b$.

d. Deduce the image of ST from the image of S and T.

e. Use the rank–nullity theorem to determine the dimension of the null space of ST.

f. Find a basis of the null space of ST.

Question 2

Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Suppose you are required to find a linear transformation that stretches every multiple of the vector v_1 by a factor 2, which fixes every point on the line $\{x = \lambda v_2 \ : \ \lambda \in \mathbb{R}\}$, and which maps the line $\{x = \lambda v_3 \ : \ \lambda \in \mathbb{R}\}$ to 0.

Show that no such linear transformation can exist.

Question 3

Find A^5 if $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$.

Question 4

Find an orthonormal basis for the subspace of $\mathbb{R}^3$ given by

$$V = \left\{ x \in \mathbb{R}^3 \,:\, 5x_1 - x_2 + 2x_3 = 0 \right\},$$

and extend it to an orthonormal basis of $\mathbb{R}^3$.

Question 5

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

a. Show that v_1 is an eigenvector of A and find its corresponding eigenvalue.

b. Given that $\lambda = 8$ is the only other eigenvalue of A: can you be certain that A can be diagonalised without doing the computations?

c. Diagonalise A by finding an invertible matrix C and a diagonal matrix Λ such that $C^{-1}AC = \Lambda$.

d. Find an orthogonal matrix S such that $S^TAS = \Lambda$.

e. For $x \in \mathbb{R}^3$, let $f(x) = x^TAx$. Write out $f(x)$.

f. Is the quadratic form f positive definite, indefinite, negative definite, or none of these?

g. Find a basis $B = \{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ such that $f(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^3$, where y_1, y_2, y_3 are the coordinates of x in the basis B.

h. Find the transition matrix from coordinates with respect to the basis B to the standard basis.

i. Let z_1 and z_2 be unit eigenvectors for each eigenvalue. Evaluate $f(x)$ at these vectors.